Godel's Incompleteness Proof | By Michel Alexis and Jan Dlabal

At the turn of the 20th century, there was a new demand for rigor within mathematics. Georg Cantor had just given a formal, rigorous definition of infinity, which until then had been poorly defined by Leibniz and Newton. Many contemporary mathematicians were inspired to rigorously redefine and prove even the most basic concepts of mathematics. This idea was exemplified by David Hilbert's program. He sought to, and challenged mathematicians across the globe, to axiomatize mathematics. This meant constructing a formal mathematical system based off of fundamental axioms, where any mathematical statement could be proved or disproved, using only this set of axioms as a basis for demonstration, and where no contradictions would arise. Such a system would be called complete and consistent. In order to accomplish this, David Hilbert encouraged mathematicians to relate all the various branches of mathematics (geometry, calculus...) to arithmetic. This meant formalizing all these branches such that, if arithmetic was shown to be a complete and consistent system, then these branches would be so too. Naturally, David Hilbert lead this program, carrying out this process for all of geometry by himself. In a couple of years, all the branches of mathematics had been related to arithmetic. All that was now left was to show that arithmetic was complete and consistent.

This task was taken up by 2 logicians, Alfred Whitehead and Bertrand Russel. In their work *Principia Mathematica*, they sought to axiomatize arithmetic into a complete and consistent mathematical system, using as axioms the most basic of logical concepts. They published their first volume of *Principia Mathematica* after 10 years of laborious work. The other volumes would not be published.

This was due to Kurt Godel's famous Incompleteness Theorems. In these theorems, Godel showed that the system of arithmetic outlined in *Principia Mathematica* was not only incomplete, but also could not confirm its own consistency. More generally, he showed that any mathematical system is fundamentally incomplete and cannot demonstrate its own consistency.

We will show the basic principles of Godel's proof, in a format that doesn't oversimplify the proof, but is still accessible to people without an advanced mathematics formation. First we will explain how he reduced the system described in *Principia Mathematica* (PM) and similar systems to a system that used only numbers, each of which had a specific meaning. Then, we will show how Godel created a mathematical statement which was undecidable within the boundaries of PM, but was nonetheless true. Finally, we will show how Godel used this statement to show that PM was both incomplete and inconsistent.

First we must define what makes up a system. A system can be thought of as of a framework for logical thought. Generally, such a system is composed of "meaningless" signs, axioms, and "grammatical" rules. Basically, one picks a set of signs (devoid of any meaning), and announces a set of axioms. To derive any and all theorems, one proceeds by taking one of the axioms, and then using the various "grammatical" rules in order to modify the axiom to obtain a theorem.

Next, we must make the distinction between mathematics and meta-mathematics. It is pretty easy to get an intuitive sense of distinction between these when looking over a few examples. As a general rule, a statement belongs to mathematics if it uses only mathematical signs, and to meta-mathematics if it uses more than that. Let's look at a few examples:

$2x = x + x$ → this statement belongs to mathematics, as it uses only mathematical signs.

x is a variable → this statement belongs to meta-mathematics, as it says something of x using not only mathematical signs.

$1+1=2$ → this statement belongs to mathematics, as it uses only mathematical signs.

the first symbol of the statement "$1+1=2$" is a 1 → this statement belongs to meta-mathematics, as it talks about mathematics using not only mathematical signs.

Now we will outline how Godel created what is called a formal calculus of mathematics. Mathematics as we know it is already very formalized: instead of relying on a natural language like English, mathematicians use specific, unambiguous signs such as “=”. Gödel decided to take this idea of formalization even further, coding mathematics and all of its statements into a purely numerical system.

To achieve this, Gödel assigned each of the signs required to construct a system like PM a “Gödel” number **(See Table 1**). Furthermore, in the construction of PM, 3 variables x, y and z were used; Gödel assigned the numbers 13, 17 and 19 respectively to each (he used prime numbers.

	Godel number	Meaning	Example
~	1	not	$\sim P$
V	2	or	P V Q
$\supset$	3	if...then...	P $\supset$ Q
$\exists$	4	There is an...	$\exists$ x (x = s0)
$=$	5	equals	s0=0+s0
0	6	zero	0
s	7	Immediate successor of	ss0=s0+s0
(	8	punctuation	
)	9	punctuation	
,	10	punctuation	
+	11	plus	$sss0=s0+s0+s0$
$\times$	12	times	$sss0=s0\times sss0$
x	13	Numerical variable	x refers to 5
y	17	Numerical variable	y refers to 4
z	19	Numerical variable	z refers to 3
P	13^2	Sentential variable	P refers to the sentence R V Q
Q	17^2	Sentential variable	Q refers to the sentence R
R	19^2	Sentential variable	R refers to the sentence ss0 = s0+ s0

Table 1: Signs and their Godel numbers

Next, Gödel defined a way through which each statement (a statement being an ensemble of elementary mathematics signs, like x=0) and combinations of statements could be assigned its own unique Gödel number. For instance, take the statement “ There is a number x such that x is the immediate successor of y.” When we translate this into mathematical signs described in Table 1, we get the statement $(\exists x)(x=sy)$. To give this statement its unique Godel number, Godel first looked at the Godel number of each individual sign. Next, Godel listed the prime numbers, beginning with 2, in ascending order. He then raised each prime number to the exponent of the corresponding sign. For example, since the Godel number of a left parenthesis is 8, and it's the first character in the sentence, we'd replace it by 2^8. This is done for each sign; and at the end, the obtained prime numbers with their exponents are all multiplied in order to get the final Godel number of the whole statement. (**See Figure 1 below).**

$$\begin{array}{cccccccccc} (& \exists & x &) & (& x & = & s & y &) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 8 & 4 & 13 & 9 & 8 & 13 & 5 & 7 & 17 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2^8 & 3^4 & 5^{13} & 7^9 & 11^8 & 13^{13} & 17^5 & 19^7 & 23^{17} & 29^9 \end{array}$$

$$\text{Godel number} = 2^8 \times 3^4 \times 5^{13} \times 7^9 \times 11^8 \times 13^{13} \times 17^5 \times 19^7 \times 23^{17} \times 29^9$$

Figure 1: translating a statement into its Godel number

Using this translation method, we can get a unique Godel number for every statement, because every positive integer has a unique prime factorization (every whole number can be written as a unique multiplication of exponents of prime numbers). As shown on **figure 2**, if we take the number 243,000,000, we can deduce the corresponding mathematical statement, $0=0$.

A	243,000,000
B	64 × 243 × 15,625
C	$2^6 \times 3^5 \times 5^6$
D	$\begin{array}{ccc} 6 & 5 & 6 \\ \downarrow & \downarrow & \downarrow \\ 0 & = & 0 \end{array}$
E	0 = 0

Figure 2: translating a Godel number into the corresponding statement

Similarly, Godel also designed a way to attribute Godel numbers to sequences of formulas (as opposed to assigning them to signs, and single formulas). This again consists of taking the Godel numbers of each statement composing the statement, and raising each of the prime numbers, in ascending order, beginning from 2 to the power of the corresponding statement's Godel number. Let's say, for example, that we want to get the Godel number for the sequence of formulas below:

$$(\ \exists \ x \ ) \ (\ x = s \ y \ )$$

$$0=0$$

We saw already that the Godel number of the first formula is $2^8 \times 3^4 \times 5^{13} \times 7^9 \times 11^8 \times 13^{13} \times 17^5 \times 19^7 \times 23^{17} \times 29^9$ (see Figure 1), and the Godel number of the second formula is $243{,}000{,}000$ (see Figure 2). The Godel number of the sequence of these two formulas would therefore be:

$$2^{\text{Godel number of first formula}} \times 3^{\text{Godel number of second formula}} = 2^{2^8 \times 3^4 \times 5^{13} \times 7^9 \times 11^8 \times 13^{13} \times 17^5 \times 19^7 \times 23^{17} \times 29^9} \times 3^{243{,}000{,}000} = \textit{large number}$$

That's great, but what's the point, you ask. Essentially, Godel developed this system in order to be able to translate meta-mathematical statements into purely mathematical ones. The importance of this will become more apparent when we explain Godel's actual proof; essentially, in order to prove the consistence/completeness of a system, we need to use the logic of the system itself, otherwise we could be using reasoning not covered by one of the axioms of the system. Therefore, since we need meta-mathematics to prove the incompleteness of mathematics, but can use only mathematical reasoning, we need a way to translate meta-mathematical reasoning into mathematical reasoning. This is precisely what Godel's numbering system allows us to do.

Let's demonstrate this by an example. Let's take the meta-mathematical statement the first character of "~ (0 = 0)" is a tilde. How would we translate this into a purely mathematical statement? The Godel number of the formula inside the statement is $2^2 \times 3^8 \times 5^6 \times 7^5 \times 11^6 \times 13^9$. Let's call this number a. We want to develop a number theoretical way of determining that ~ is the first sign of our initial statement. If ~ is the first sign of our initial statement, then 2^1 must be a factor of a, because 1 is the Godel number for the tilde, and the tilde has to be the first character (so the 1 is the exponent of the 2). Moreover, for ~ to be the first sign of our statement, any other power of 2 other than 2 to the power of 1 cannot be a factor of a (otherwise the first sign wouldn't be a tilde). In other words, 2^1 must be a factor of a but 2^2 cannot. This translates into the following statement in PM:

$(\exists z)(\underbrace{sss...sss}_{\text{a times}}0 = z \times ss0) \cdot \sim(\exists z)(\underbrace{sss...sss}_{\text{a times}}0 = z \times (ss0 \times ss0))$, where the long numeral consists of a number of 's'.

This complicated-looking formula means "There exists a number z such that a is equal to z times 2 and there doesn't exist a number z such that a is equal to z times 2 times 2." Although not impressive at first, this statement is an example of how PM can make statements about its own formulas. This is an example of how Godel's arithmetization of PM allows PM to 'talk' about itself, and how we can translate meta-mathematical statements into purely mathematical ones.

Moreover, the formula at which we just arrived is guaranteed to be a theorem of PM by the Correspondence Lemma, because the basis of it, "x is a factor of y," is what we call a primitive recursive statement. Proving that a statement is primitive recursive is rather complicated, but as an example, all "obvious" statements such as $1+1=2$, $4\times5=20$, etc., are primitive recursive statements.

Now, let's imagine a more complex type of meta-mathematical statement. "The sequence of formulas with Godel number x is a proof (in PM) of the formula with Godel number z." Let's imagine a sequence of n statements with Godel numbers $a_1, a_2, \ldots, a_{n-1}, a_n$, which are a proof of the statement with Godel number a_n . The Godel number k of this proof is $k = 2^{a_1} \times 3^{a_2} \times 5^{a_3} \times \ldots \times P_n^{a_n}$, where P_n is the nth prime. After a little reflection, we can see that there is a definite (though not at all simple) relationship between k and a_n . Let's note this numerical relationship between two numbers a and b dem (a;b). In his paper, Godel shows that dem(a;b) is a primitive recursive relationship. Therefore, according to the Correspondence Lemma, there is also a formula within PM expressing this relationship using formal notation and signs. We will denote this formula as Dem (a;b). **Note:** dem(a;b) is an actual numerical relationship, whereas Dem(a;b) is strictly speaking a meaningless string of signs using the notation of PM. However, the existence of Dem (x;y) tells us that assertions like "such and such proves such and such, according to PM" are also theorems of PM (the importance of this will become evident when we will describe Godel's actual proof).

We need to introduce one last idea before we can take on Godel's actual proof. This idea is relatively simple to understand; basically, Godel pointed out that any formula has a Godel number attached to it, and that this number can be substituted for a variable inside the original formula in order to get a new Godel number. For example, we already saw, in Figure 1, that the formula $(\exists x)(x=sy)$ has a Godel number, let's call it a . Now, we know we may replace the variable y by any number, so we could replace it with a . We'd get the formula : $(\exists x)(x=sss...sss0)$, with $a+1$ copies of s. Although it is certainly not simple, we can see that we could perfectly calculate the Godel number of the new statement, and we could also develop a function that would give the new Godel number as a function of the old one. Godel decided to denote this function by $\text{sub}(x, 17, x)$. In plain English, this number represents the Godel number of the formula obtained by taking the formula with Godel number x and wherever there are occurrences of the variable y (hence the 17), substituting with the number x .

Next, Godel has also shown that this function is a primitive recursive; thus there is also a theorem in PM, that exactly mirrors the function $\text{sub}(x, 17, x)$; this formal expression within PM will be henceforth be referred to as $\text{Sub}(x, 17, x)$.

Godel's Proof

Now, let us explain the heart of Godel's new argument. We will first explain how Godel constructed a formula G with godel number g representing the meta-mathematical statement: “the formula with godel number g is not demonstrable using the rules of PM.” Next, we will outline how Godel showed that G is demonstrable if and only if ~G is demonstrable. This means that PM would thus be inconsistent. Then, Godel showed that although G isn't formally demonstrable, it is a true arithmetical formula, in the sense that it claims a certain arithmetical property which no integer possesses. The logical conclusion is that since G is true and formally undecidable within PM, then PM must be incomplete. He shows that even if additional axioms were added to make G formally demonstrable within a new type of PM, then a new formula G' could be constructed similarly as G was with all of the same properties. Finally, Godel outlined how to create another formula A representing the meta-mathematical statement “PM is consistent.” He then demonstrated the formula $A \supset G$ (meaning if A is true then G is true). He also showed that A isn't formally demonstrable within PM. This whole chain of reasoning shows that there is no way to logically demonstrate the consistency of PM within PM itself.

Now we will explain each of the logical steps listed above in greater detail.

Let us imagine the following formula of PM $(\exists x)\, Dem(x\,;z)$, with Dem (x;y) being the relation we defined earlier. In plain English this formula actually means “there exists a sequence of formulas with Godel number x that constitutes a demonstration of the formula with Godel number z.” Negating the formula, we get $\sim(\exists x)\, Dem(x\,;z)$ which means there isn't a sequence of formulas which constitutes a proof of the formula with Godel number z. In other words the formula with Godel number z isn't demonstrable. Next, Godel substitutes z with $z = \text{sub(y,17,y)}$, creating this formula $\sim(\exists x)\, Dem(x\,,\text{sub(y, 17, y)})$. This formula does indeed belong to PM; it has a specific Godel number assigned to it. Let's call this number n. Godel now takes all instances of y and replaces it by n. We're left with $\sim(\exists x)\, Dem(x\,,\text{sub(n, 17, n)})$, which we shall call formula G. In meta-mathematics, this statement means that the formula with Godel number sub(n, 17, n) is not demonstrable within PM. However, one will notice that to obtain the formula above, one needs to take the formula $\sim(\exists x)\, Dem(x\,,\text{sub(y, 17, y)})$ and substitute y for n. After some reflection from the reader (go back to the definition of the “sub” function if you must), we can see that the Godel number g of formula G is actually sub(n , 17, n). Thus, we have successfully constructed the formula with Godel number g which mirrors the meta-mathematical statement the formula with Godel number g is not formally demonstrable. In other words, the formula G states that it itself is not formally demonstrable.

Let's suppose that the formula G is actually demonstrable within PM. Then there must be a sequence of formulas demonstrating G. Thus, the relationship $dem(k\,,z)$ must be verified, in the case where k is the Godel number of the sequence of formulas proving G, of Godel number z . Substituting z with $\text{sub}(n\,;17\,;n)$, we have the following true arithmetical relationship: $dem(k\,,\text{sub (n, 17, n)})$. Since this formula represents a primitive recursive relationship, then as we showed earlier $Dem(k\,,\text{sub (n, 17, n)})$ must be a theorem of PM. Moreover, a rule of inference of PM states that if we have a property P(k) which is true, then we can also write $(\exists x)\, P(x)$. This allows us to write $(\exists x)\, Dem(x\,,\text{sub (n, 17, n)})$. We realize that this formula is actually the formal negation of G. Thus, if G is demonstrable, then ~G is also demonstrable within PM. Godel also demonstrated in his paper that if ~G were demonstrable, then so would G be demonstrable. The only way for this to be possible is if PM is in fact inconsistent. However, if PM is consistent, then neither G nor ~G is demonstrable; G must be an undecidable statement of PM.

The question we face right now is whether G or ~G is true. With meta-mathematical reasoning, we can see that G is true (if ~G were true, then G, the formula which clearly states it is not demonstrable, would be demonstrable). Moreover, since G is undecidable in PM, it has no proof within

PM, which is what G asserts. G is a true statement. G is true arithmetical formula.

Thus, PM must be incomplete. But suppose one were to add G as an axiom to PM. Godel showed that by creating a new, more complex relation $dem'(x, z)$ (as demonstrability would have been changed in this new system), one could still create a new statement G' with the same properties as G. The new system would also be incomplete. He even showed that the same sort of statement could be showed no matter what new axioms were added to PM. He proved that PM is fundamentally inconsistent.

The last part of Godel's proof consists of showing that PM cannot formally demonstrate its own consistency. He does this by first taking the meta-mathematical statement “If PM is consistent, then it is incomplete” and formalizing the notion. The first half of this statement “PM is consistent,” has been represented by all of the reasoning we have done thus far on G and ~G. As we showed earlier, PM is consistent if and only if G is not demonstrable. To put this generally, PM is consistent if and only if there exists a formula (Godel number y) which cannot be demonstrated by any sequence of formulas (Godel number x) within PM. The formal way of writing this is $(\exists y) \sim Dem(x, y)$. We will call this formula A. The latter half of the statement we were analyzing, “it [PM] is incomplete,” is the equivalent of having a true formula of PM which is not demonstrable. This is the case for G. Mirroring “If PM is consistent, then it is incomplete” in PM gives $(\exists y) \sim Dem(x, y) \supset \sim (\exists x) Dem(x, _{(n, 17, n)})$, or more simply put $A \supset G$ (where $\supset$ means if...then....).

Now, Godel shows that A is in fact not formally demonstrable. Suppose it were. Then, since $A \supset G$ is demonstrable, G would also have to be demonstrable. But we showed earlier that G could not be demonstrable if PM were complete. Thus, A is not demonstrable within PM. Thus, Godel shows that if PM is consistent, PM cannot demonstrate its own consistency. This is not to mean that we can't prove its consistency through meta-mathematics, but simply that any system designed like PM cannot confirm its own consistency.

Godel demonstrated that mathematical systems like PM and all others similar to it which base themselves on arithmetic are not only fundamentally incomplete, but cannot affirm their consistency.

We have just shown that mathematics is an incomplete, and therefore limited, system. What does this tell us about the state of mathematics as a science, a source of objective knowledge? Every science is somehow limited in its scope, and only when we know the limit of the scope of the subject is the subject defined most clearly, and therefore most useful to us. Moreover, each science is most precise and powerful when it explicitly defines its limits; it's important to know in which situation a tool is most useful, and in which situations it is inaccurate and worthless. In other words, we could say that a science, system, or theory is most complete when we know its limits. In this aspect, isn't then mathematics the most complete science, since it can describe its limits precisely by itself, unlike other sciences, which need another external science to expose its own limits? Indeed, the limits of Newtonian mechanics were defined and understood only once Einstein wrote his papers on Special and General Relativity. Since mathematics can define its own limits independently, it therefore stands to reason that mathematics is among the most powerful of sciences.